

Violating the Ingleton Inequality With Finite Groups

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Abstract—It is well known that there is a one-to-one correspondence between the entropy vector of a collection of n random variables and a certain group-characterizable vector obtained from a finite group and n of its subgroups [1]. However, if one restricts attention to abelian groups then not all entropy vectors can be obtained. This is an explanation for the fact shown by Dougherty et al [2] that linear network codes cannot achieve capacity in general network coding problems (since linear network codes form an abelian group). All abelian group-characterizable vectors, and by fiat all entropy vectors generated by linear network codes, satisfy a linear inequality called the Ingleton inequality. In this paper, we study the problem of finding non-abelian finite groups that yield characterizable vectors which violate the Ingleton inequality. Using a refined computer search, we find the symmetric group S_5 to be the smallest group that violates the Ingleton inequality. Careful study of the structure of this group, and its subgroups, reveals that it belongs to the Ingleton-violating family $PGL(2, p)$ with primes $p \geq 5$, i.e., the projective group of 2×2 nonsingular matrices with entries in \mathbb{F}_p . This family of groups is therefore a good candidate for constructing network codes more powerful than linear network codes.

I. INTRODUCTION

Let $\mathcal{N} = \{1, 2, \dots, n\}$, and let X_1, X_2, \dots, X_n be n jointly distributed discrete random variables. For any nonempty set $\alpha \subseteq \mathcal{N}$, let X_α denote the collection of random variables $\{X_i : i \in \alpha\}$, with joint entropy $h_\alpha \triangleq H(X_\alpha) = H(X_i; i \in \alpha)$. We call the ordered real $(2^n - 1)$ -tuple $(h_\alpha : \emptyset \neq \alpha \subseteq \mathcal{N}) \in \mathbb{R}^{2^n - 1}$ an entropy vector. The set of all entropy vectors derived from n jointly distributed discrete random variables is denoted by Γ_n^* . It is not too difficult to show that the closure of this set, i.e., $\overline{\Gamma_n^*}$, is a *convex cone*.

The set $\overline{\Gamma_n^*}$ figures prominently in information theory since it describes the possible values that the joint entropies of a collection of n discrete random variables can obtain. From a practical point of view, it is of importance since it can be shown that the capacity region of any arbitrary multi-source multi-sink *wired* network, whose graph is acyclic and whose links are discrete memoryless channels, can be obtained by optimizing a linear function of the entropy vector over the convex cone $\overline{\Gamma_n^*}$ and a set of linear constraints (defined by the network) [3], [4]. Despite this importance, the entropy region $\overline{\Gamma_n^*}$ is only known for $n = 2, 3$ random variables and remains unknown for $n \geq 4$ random variables. Nonetheless, there are important connections known between $\overline{\Gamma_n^*}$ and matroid theory (since entropy is a submodular function and therefore

somehow defines a matroid) [5], determinantal inequalities (through the connection with Gaussian random variables) [6], and quasi-uniform arrays [7]. However, perhaps most intriguing is the connection to finite groups which we briefly elaborate below.

A. Groups and Entropy

Let G be a finite group, and let G_1, G_2, \dots, G_n be n of its subgroups. For any nonempty set $\alpha \subseteq \mathcal{N}$, the group $G_\alpha \triangleq \bigcap_{i \in \alpha} G_i$ is a subgroup of G . Let $|K|$ be the order (cardinality) of a group K , and define $g_\alpha \triangleq \log \frac{|G|}{|G_\alpha|}$. We call the ordered real $(2^n - 1)$ -tuple $(g_\alpha : \emptyset \neq \alpha \subseteq \mathcal{N}) \in \mathbb{R}^{2^n - 1}$ a (finite) group characterizable vector. Let Υ_n be the set of all group characterizable vectors derived from n subgroups of a finite group.

The major result shown by Chan and Yeung in [1] is that $\overline{\Gamma_n^*} = \text{cone}(\Upsilon_n)$, i.e., the closure of Γ_n^* is the same as the closure of the cone generated by Υ_n . In other words, every group characterizable vector is an entropy vector, whereas every entropy vector is arbitrarily close to a scaled version of some group characterizable vector.

To show that every group characterizable vector is an entropy vector [1] gives the following construction. Let Λ be a random variable uniformly distributed on the elements of G . Now for $i = 1, \dots, n$ define $X_i = \Lambda G_i$ (the left coset of Λ in G w.r.t. the subgroup G_i). Then a simple calculation shows that $h_\alpha = \log \frac{|G|}{|G_\alpha|} = g_\alpha$, implying that every group-characterizable vector is an entropy vector. Showing the other direction, i.e., that every entropy vector is arbitrarily close to a scaled version of a group-characterizable vector is more tricky (the interested reader may consult [1] for the details). Here we shall briefly describe the intuition.

Consider a random variable X_1 with alphabet size N and probability mass function $\{p_i, i = 1, \dots, N\}$. Now if we make T copies of this random variable to make sequences of length T , the entropy of X_1 is roughly equal to the logarithm of the number of typical sequences. These are sequences where X_1 takes its first value roughly Tp_1 times, its second value roughly Tp_2 times and so on. Therefore assuming that T is large enough so that the Tp_i are close to integers (otherwise, we have to round things) we may roughly write

$$H(X_1) \approx \frac{1}{T} \log \begin{pmatrix} T & & & \\ Tp_1 & Tp_2 & \dots & Tp_{N-1} & Tp_N \end{pmatrix},$$

where the argument inside the log is the usual multinomial coefficient. Written in terms of factorials this is

$$H(X_1) \approx \frac{1}{T} \log \frac{T!}{(Tp_1)!(Tp_2)! \dots (Tp_N)!}. \quad (1)$$

If we consider the group G to be the symmetric group S_T , i.e., the group of permutations among T objects, then clearly $|G| = T!$. Now partition the T objects into N sets each with Tp_1 to Tp_N elements, respectively, and define the group G_1 to be the subgroup of S_T that permutes these objects *while respecting the partition*. Clearly, $|G_1| = (Tp_1)!(Tp_2)! \dots (Tp_N)!$, which is the denominator in (1). Thus, $H(X_1) \approx \frac{1}{T} \log \frac{|G|}{|G_1|}$, so that the entropy $h_{\{1\}}$ is a scaled version of the group-characterizable $g_{\{1\}}$. This argument can be made more precise and can be extended to n random variables—see [1] for the details. We note, in passing, that this construction often needs T to be very large, so that the group G and the subgroups G_i are huge.

B. The Ingleton Inequality

As mentioned earlier, entropy satisfies submodularity and therefore, with some care, defines a matroid. Matroids are defined by a ground set and a rank function, defined over subsets of the ground set, that satisfies submodularity. They were defined in a way to extend the notion of a collection of vectors (in some vector space) along with the usual definition of the rank. A matroid is called *representable* if its ground set can be represented as a collection of vectors (defined over some finite field) along with the usual rank function. Determining whether a matroid is representable or not is, in general, an open problem.

Let $n = 4$, $\mathcal{N} = \{1, 2, 3, 4\}$. In 1971 Ingleton showed that the rank function $x_{\{ \cdot \}}$ of any representable matroid must satisfy the inequality [8]

$$x_{12} + x_{13} + x_{14} + x_{23} + x_{24} \geq x_1 + x_2 + x_{34} + x_{123} + x_{124} \quad (2)$$

where for simplicity we write x_{ij} and x_{ijk} for $x_{\{i,j\}}$ and $x_{\{i,j,k\}}$, respectively. However, it turns out that there are entropy vectors that violate the Ingleton inequality [9], so that entropy is generally not a representable matroid. Using non-representable matroids, [2] constructs network coding problems that cannot be solved by linear network codes (since linear network codes are, by definition, representable).

As $\overline{\Gamma}_n^* = \overline{\text{cone}(\Upsilon_n)}$, we know there must exist finite groups, and corresponding subgroups, such that their induced group-characterizable vectors violate the Ingleton inequality. In [10] it was shown that abelian groups cannot violate the Ingleton inequality, thereby giving an alternative proof as to why linear network codes cannot achieve capacity on arbitrary networks—they form an abelian group. So we need to focus on non-abelian groups and their connections to nonlinear codes.

Finally, we remark that, in the context of finite groups, the Ingleton inequality can be rewritten as

$$|G_1||G_2||G_{34}||G_{123}||G_{124}| \geq |G_{12}||G_{13}||G_{14}||G_{23}||G_{24}| \quad (3)$$

C. Discussion

Since we know of distributions whose entropy vector violates the Ingleton inequality, we can, in principle, construct finite groups whose group-characterizable vectors violate Ingleton. Two such distributions are Example 1 in [11], where the underlying distribution is uniform over 7 points and the random variables correspond to different partitions of these seven points, and the example on page 1445 of [12], constructed from finite projective geometry and where the underlying distribution is uniform over $12 \times 13 = 156$ points. Unfortunately, constructing groups and subgroups for these distributions using the recipe of section I-A results in $T = 29 \times 7 = 203$ and $T = 23 \times 156 = 3588$, which results in groups of size $203!$ and $3588!$, which are too huge to give us any insight whatsoever.

These discussions lead us to the following questions.

- 1) Could the connection between entropy and groups be a red herring? Are the interesting groups too large to give any insight into the problem (e.g., the conditions for the Ingleton inequality to be violated)?
- 2) What is the smallest group with subgroups that violates the Ingleton inequality? Does it have any special structure?
- 3) Can one construct network codes from such Ingleton-violating groups?

In this paper we address the first two questions. We identify the smallest group that violates the Ingleton inequality—it is the symmetric group S_5 , with 120 elements. Through a thorough investigation of the structure of its subgroups we conclude that it belongs to the family of groups $PGL(2, p)$, with p a prime greater than or equal to 5. ($PGL(2, 5)$ is isomorphic to S_5 .)¹ We therefore believe that the connection to groups is not a red herring and that there may be some benefit to it.

The explicit nature of $PGL(2, p)$ may lend itself to effective network codes. We only mention that non-abelian groups allow for much more flexibility in the design of codes. For example, if the incoming messages to a node in the network, a and b , say, are elements from a nonabelian group then the operations a^2b , aba , ba^2 , say, can potentially all correspond to different elements in the group, whereas in the abelian case they all coincide with a^2b . Therefore nodes in a network will have much more choices in terms of what to transmit on their outgoing edges—and this should, ostensibly, be what allows one to achieve capacity. The drawback is, of course, that decoding becomes more complicated than solving a system of linear equations.

We shall not say anymore about codes. What we will do in the remainder of the paper is to describe how we found the smallest Ingleton-violating group and how we uncovered its structure. This required the identification of conditions beyond being abelian that force a group to respect Ingleton. It also required a deep study of the 120 element group that we found via computer search. We now present the details.

¹ $GL(2, p)$ is the general linear group of invertible 2×2 matrices with entries in \mathbb{F}_p . $PGL(2, p)$ is the *projective* general linear group, where proportional matrices in $GL(2, p)$ are all mapped to the same element.

II. NOTATION

We use the following abstract algebra notations throughout this paper:

$ G $	the order of group G .
$G \cong H$	the group G is isomorphic to the group H .
$H \leq G, H < G$	H is a subgroup of G , and a proper subgroup of G .
$H \trianglelefteq G$	H is a normal subgroup of G .
G/H	the set of all left cosets of subgroup H in G . When $H \trianglelefteq G$, G/H is a group. (Factor or quotient group)
$ g $	the order of element g = smallest positive integer m s.t. $g^m = 1$.
x^g	the conjugate of element x by element g in G : $x^g = g^{-1}xg$. (No confusion with the powers of x as g is an element of G .)
X^g	the conjugate of subset X by element g in G : $X^g = \{x^g : x \in X\}$.
HK	the “set product” of $H, K \subseteq G$: $HK = \{hk : h \in H, k \in K\}$.
$H \rtimes K$	the semidirect product of groups H and K .
$\langle g_1, \dots, g_m \rangle, \langle S \rangle$	the group generated by the elements g_1, \dots, g_m , and by the set S .
$G = \langle S R \rangle$	$\langle S R \rangle$ is a presentation of G . S is a set of generators of G , while R is a set of relations G should satisfy.
1	the natural number “1”, identity element of a group, or the trivial group. The meaning should be clear in different contexts with no confusion.
\mathbb{Z}_n	the integers modulo $n \cong$ the cyclic group of order n .
S_n	the symmetric group of degree n = all permutations on n points.
D_{2n}	the dihedral group of order $2n$.
\mathbb{F}_q	the finite field of q elements.
$\mathbb{Z}_n^\times, \mathbb{F}_q^\times$	the multiplicative group of units of \mathbb{Z}_n , and of \mathbb{F}_q . \mathbb{F}_q^\times = all nonzero elements of \mathbb{F}_q .
$GL(n, q)$	the general linear group of all invertible $n \times n$ matrices with entries from \mathbb{F}_q . The identity element for $GL(n, q)$ is usually denoted by I = identity matrix.
$PGL(n, q)$	the projective general linear group = $GL(n, q)/V$, where V = all nonzero scalar matrices = $\{\alpha I : \alpha \in \mathbb{F}_q^\times\}$.

III. COMPUTER SEARCH AND SOME NEGATIVE CONDITIONS

Designing a small admissible structure for the group G and its subgroups without an existing Ingleton-violating instance is very difficult, so we use computer programs to search for a small instance. We use the GAP system [13] to search its

“Small Group” library, which contains all finite groups of order less than or equal to 2000 except 1024. We pick a group in this library, find all its subgroups, then test Ingleton inequality for all 4-combinations of these subgroups. This is a tremendous task, as there are already more than 1000 groups of order less than or equal to 100, each of which might have hundreds of subgroups (some even have more than 1000).

It was therefore extremely critical to prune our search. In fact, we used the following “negative conditions”, each of which guarantees that Ingleton is never violated.

Condition 1: G is abelian. [10]

Condition 2: $G_i \trianglelefteq G, \forall i$. [14]

Condition 3: $G_1G_2 = G_2G_1$, or equivalently $G_1G_2 \leq G$.

Proof: (sketch) Construct random variables X_i 's from uniformly distributed Λ on G as in Section I-A. As $G_{1;2} \triangleq G_1G_2 \leq G$, we can similarly construct random variable $X_{1;2} = \Lambda_{G_{1;2}}$. Note that $|G_{1;2}| = |G_1||G_2|/|G_{12}|$, $H(X_{1;2}|X_1) = H(X_{1;2}|X_2) = 0$ as $G_1, G_2 \leq G_{1;2}$. Similar to the proof of Condition 2 in [14], we use the following information inequality in [15]:

$$2H(E|A) + 2H(E|B) + I(A; B|C) + I(A; B|D) + I(C; D) \geq H(E).$$

Plugging in $A = X_1, B = X_2, C = X_3, D = X_4$ and $E = X_{1;2}$ one can easily deduce Ingleton inequality. ■

Remark 1: Condition 2 subsumes Condition 1, while Condition 3 subsumes Condition 2.

Remark 2: In the proof of condition 3 we used the aforementioned group-entropy relation to translate the problem to the entropy domain. We shall prove most of the conditions in this manner.

Observe that the Ingleton inequality has symmetries between subscripts 1 and 2 and between 3 and 4, i.e. if we interchange the subscripts 1 with 2 or 3 with 4, the inequality stays the same. Thus if we prove some conditions for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$, we automatically get conditions for all $(i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$. So without loss of generality, we will just prove conditions for $i \in \{1, 3\}$, or $(i, j) \in \{(1, 2), (1, 3), (3, 1), (3, 4)\}$ when these symmetries apply.

Condition 4: $G_i = 1$ or G , for some i .

Proof: For $i = 1$, either would imply $G_1G_2 = G_2G_1$ in Condition 3. For $i = 3$, $|G_3| = 1$ implies that the Ingleton inequality becomes $|G_1||G_2||G_{124}| \geq |G_{12}||G_{14}||G_{24}|$, which clearly follows from $|G_1||G_{124}| \geq |G_{12}||G_{14}|$ (implied by submodularity of entropy) and $|G_2| \geq |G_{24}|$. ■

Condition 5: $G_i = G_j$ for some distinct (i, j) .

Proof: (sketch) For $(i, j) = (1, 2)$, use $G_1G_2 = G_2G_1$ in Condition 3. For $(1, 3)$ and $(3, 4)$, the argument is similar to that of the previous condition. ■

Condition 6: $G_{12} = 1$.

Proof: Realize that Ingleton inequality for entropy vectors can be rewritten as

$$r_{13,14} + r_{23,24} + r_{134,234} - r_{123,124} \geq 0, \quad (4)$$

where $r_{\alpha,\beta} \triangleq h_\alpha + h_\beta - h_{\alpha \cap \beta} - h_{\alpha \cup \beta}$ for $\emptyset \neq \alpha, \beta \subseteq \mathcal{N}$. (e.g., $r_{134,234} = h_{134} + h_{234} - h_{34} - h_{1234}$.) By submodularity, all $r_{\alpha,\beta} \geq 0$. If $G_{12} = 1$, then $r_{123,124} = 0$ and (4) holds. ■

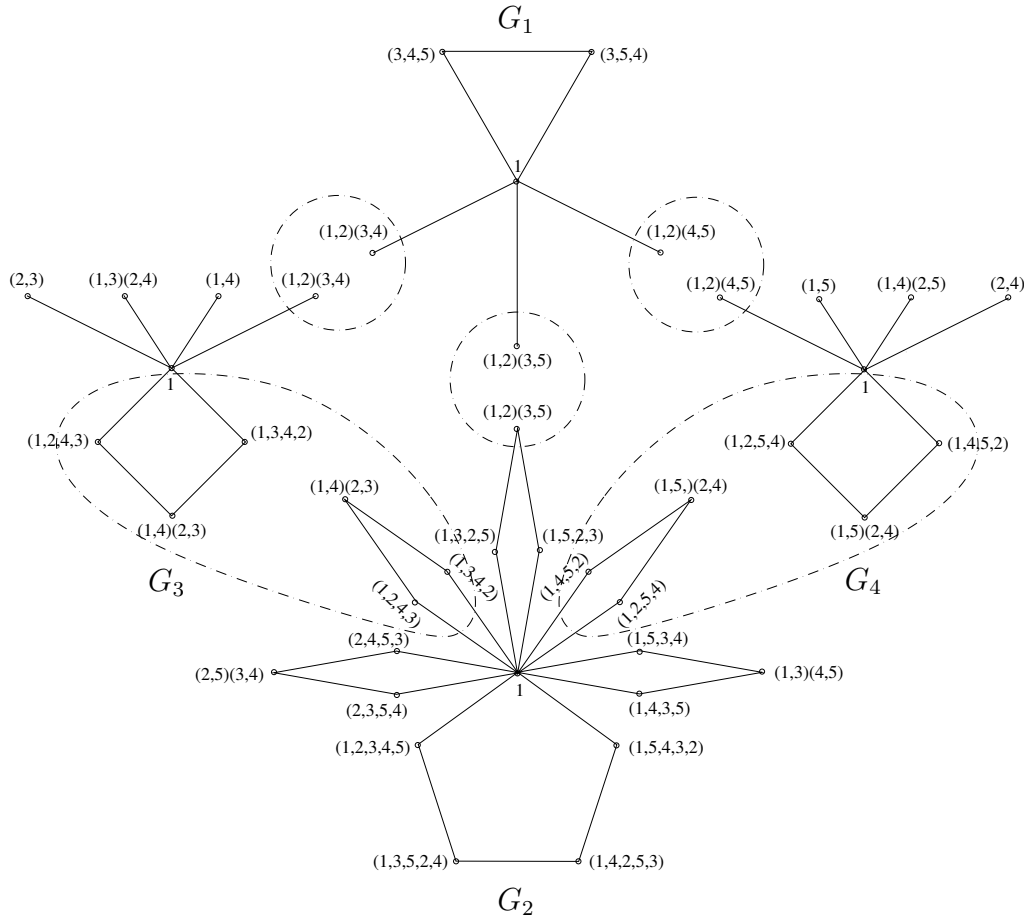


Fig. 1. The cycle graph of the Ingleton violating subgroups of S_5

Condition 7: $G_i \leq G_j$ for some distinct (i, j) .

Proof: (sketch) $(i, j) = (1, 2)$ implies $G_1 G_2 = G_2 G_1$. $(1, 3)$ implies $r_{123,124} = 0$ in (4). $(3, 1)$ implies $r_{123,234} = 0 \Rightarrow r_{123,234} \leq r_{12,24} \Rightarrow r_{123,124} \leq r_{23,24} \Rightarrow (4)$ holds. For $(3, 4)$, rewrite $h_{13} = h_{134}$, $h_{23} = h_{234}$, $h_{123} = h_{1234}$, then use submodularity and non-negativeness of entropy. ■

Remark 3: Conditions 6 and 7 were first pointed out to us by Prof. M. Aschbacher using group theoretic techniques. The proof presented above is based on the submodularity and non-negativity of entropy.

Remark 4: Conditions 1, 3 and 6 are crucial in our searching program, as they appear in the outer searching loops and can reduce a large amount of work.

IV. THE SMALLEST VIOLATION INSTANCE AND ITS STRUCTURE

Using GAP we found the smallest group that violates Ingleton is $G = S_5$. There are 60 sets of violating subgroups if we eliminate the influence of subscript symmetries. Furthermore, these 60 sets of subgroups are all conjugates of each other. Thus in terms of group structure, these instances are virtually the same. We list below some information from GAP about one representative: (the permutations are written in cycle notation, e.g. $(3, 4, 5)$ is the permutation that maps

element 3 to 4, element 4 to 5, and element 5 to 3).

$G_1 = \langle (3, 4, 5), (1, 2)(4, 5) \rangle$	$\cong S_3 \cong D_6$	$ G_1 = 6$
$G_2 = \langle (1, 2, 3, 4, 5), (1, 4, 3, 5) \rangle$	$\cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$ G_2 = 20$
$G_3 = \langle (2, 3), (1, 3, 4, 2) \rangle$	$\cong D_8$	$ G_3 = 8$
$G_4 = \langle (2, 4), (1, 2, 5, 4) \rangle$	$\cong D_8$	$ G_4 = 8$
$G_{12} = \langle (1, 2)(3, 5) \rangle$	$\cong \mathbb{Z}_2$	$ G_{12} = 2$
$G_{13} = \langle (1, 2)(3, 4) \rangle$	$\cong \mathbb{Z}_2$	$ G_{13} = 2$
$G_{14} = \langle (1, 2)(4, 5) \rangle$	$\cong \mathbb{Z}_2$	$ G_{14} = 2$
$G_{23} = \langle (1, 3, 4, 2) \rangle$	$\cong \mathbb{Z}_4$	$ G_{23} = 4$
$G_{24} = \langle (1, 2, 5, 4) \rangle$	$\cong \mathbb{Z}_4$	$ G_{24} = 4$
$G_{34} = 1$		$ G_{34} = 1$
$G_{123} = 1$		$ G_{123} = 1$
$G_{124} = 1$		$ G_{124} = 1$

As $|G_1||G_2||G_{34}||G_{123}||G_{124}| = 120 < 128 = |G_{12}||G_{13}||G_{14}||G_{23}||G_{24}|$, Ingleton is violated. Also $G_1 - G_4$ generate $G = S_5$.

To illustrate the structure of these subgroups, we use the group cycle graph. See Fig.1, where the dash-dotted lines denote the pairwise intersections of subgroups excluding identity. From the cycle graph we can obtain more structural information which GAP does not show us directly. First, not only is G_2 a semidirect product of two cyclic groups $\langle (1, 2, 3, 4, 5) \rangle \cong \mathbb{Z}_5$ and $\langle (1, 4, 3, 5) \rangle \cong \mathbb{Z}_4$ (in particular, it's metacyclic), but also $G_2 \setminus \langle (1, 2, 3, 4, 5) \rangle \cup \{1\}$ is the union of subgroups which are all isomorphic to $\langle (1, 4, 3, 5) \rangle$

(actually they are all conjugates of $\langle(1, 4, 3, 5)\rangle$) and have trivial pairwise intersections. (In this case we say G_2 has a “flower” structure.) Second, G_4 is the conjugate of G_3 by $(3, 4, 5)$ in G_1 . In particular, $(1, 3, 4, 2)^{(3, 4, 5)} = (1, 4, 5, 2) = (1, 2, 5, 4)^{-1}$.

As these subgroups are represented in permutations, it is not easy either to construct a code from them, or to extend them to a family of violations. Naturally one may try S_6 with similar subgroups, but unfortunately they do not work. A better way to extract the structural information and extend the subgroups to a family of (possible Ingleton-violating) instances, is through the abstract presentation of groups. It might still be difficult to see concrete group elements or to prove the structure is successfully extended, however, we can feed the (extended) presentation to GAP and it might determine a concrete isomorphic group, which preserves the structure of violation.

Observe that $|G_{23}| = |G_{24}| = 4$ contribute most to the *RHS* of (3), we may try to let the “petals” of G_2 (conjugates of $\langle(1, 4, 3, 5)\rangle$) grow while keep other structures fixed. (This is a little conservative, but it is the only successful extension according to our GAP trials. For example, one may try to extend G_1 at the same time, but the structure of G_3 and G_4 usually collapse.) As G_2 plays the most important role in the violation, we can start from extending the flower structure of G_2 . Specifically, we may assume that $G_2 = \langle a, b \rangle$ which has a normal subgroup $N = \langle a \rangle \cong \mathbb{Z}_n$, as well as a subgroup $H = \langle b \rangle \cong \mathbb{Z}_m$, for some generators a, b and integers m, n . This gives us a presentation

$$G_2 = \langle a, b \mid a^n = b^m = 1, a^b = a^s \rangle \quad (5)$$

for some $0 < s < n$. In order to violate Ingleton as much as possible, we may wish n to be small while m large. However, the flower structure of G_2 may limit the choices of n and m . First of all, for this presentation to be a semidirect product, we need $s^m \equiv 1 \pmod{n}$ (see [16], 5.4). In this case a, b have order n, m respectively, $|G_2| = mn$, $H \cap N = 1$, $s \in \mathbb{Z}_n^\times$ with $|s| \mid m$, also $(a^i)^{b^k} = a^{is^k}$ for any integers i and k . Moreover, we need $G_2 \setminus N \cup \{1\}$ to be the union of groups which are all isomorphic to H with trivial pairwise intersections.

One possible way to achieve this is to restrict $H^{g_1} \cap H^{g_2} = 1 \ \forall g_1 \neq g_2 \in N$, as in our original construction. This is equivalent to $H^g \cap H = 1, \forall g \in N \setminus \{1\}$. If this is the case, there would be $|N| = n$ “petals” of size m in G_2 and the total number of nonidentity elements would equal $n(m-1) = nm - n = |G_2 \setminus N|$, so indeed the flower structure would be achieved.

Pick two arbitrary nonidentity elements $h_1 = b^l \in H$, $h_2 = (b^k)^{a^i} \in H^{a^i}$ for some $0 < k, l < m$, $0 < i < n$. $h_1 = h_2 \Leftrightarrow a^{-i}b^ka^i = b^l \Leftrightarrow a^{-i}(a^i)^{b^{-k}}b^k = b^l \Leftrightarrow a^{-i}a^{is^{-k}} = b^{l-k} \Leftrightarrow a^{(s^{-k}-1)i} = b^{l-k}$. As $H \cap N = 1$, this is equivalent to $a^{(s^{-k}-1)i} = b^{l-k} = 1$. i.e. $l = k, n \mid (s^{-k}-1)i$. To guarantee $H^{a^i} \cap H = 1$ for any $0 < i < n$, we must have $m \leq |s|$. Otherwise we can just choose $0 < k = |s| < m$, then $s^{-k} \equiv 1 \pmod{n} \Rightarrow n \mid (s^{-k}-1)i$ for any i , and we find a nonidentity element $h_1 = b^k = h_2 = (b^k)^{a^i}$ in $H^{a^i} \cap H$. So

$m \leq |s|$ with $|s| \mid m \Rightarrow m = |s|$. In particular, $m \leq \phi(n) < n$, where $\phi(n) = |\mathbb{Z}_n^\times|$ is Euler’s totient function.

For m to be as large as possible, s should be a primitive root modulo n , which makes $m = \phi(n)$. Furthermore, if we choose $n = p$ for some prime $p > 2$, then $m = \phi(p) = p-1$ is relatively “maximized”. (We need $p > 2$ for the petals not to collapse.) Also in this situation if we let $0 < k < m = |s|$, $0 < i < n = p$, then $n \mid (s^{-k}-1)i$ requires $p \mid i$ or $p \mid (s^{-k}-1)$. As p is prime, $p \nmid i$, so $p \mid (s^{-k}-1) \Rightarrow s^{-k} \equiv 1 \pmod{p} \Rightarrow |s| \mid k$. But $0 < k < |s|$, contradiction. So actually we have $H^g \cap H = 1, \forall g \in N$ and the flower structure is realized.

Now assume n, m and s are as above. The next step is to extend presentation (5) to the whole group G generated by G_1 — G_4 . Consider the dihedral groups G_3 and G_4 . The subgroups of rotations are just H^{a_3} and H^{a_4} respectively, for some $a_3 = a^{k_3}, a_4 = a^{k_4} \in N$. Also G_3 and G_4 each shares one element of reflection with the dihedral group G_1 , while the remaining reflection of G_1 is just $(b^{\frac{p-1}{2}})^{a_1}$ in G_2 , for some $a_1 = a^{k_1} \in N$. Thus if we can determine the generator of the subgroup of rotations of G_1 , then all elements of G_1 — G_4 are determined. In other words, if we introduce an element c as the generator of rotations of G_1 , then all elements from G_1 — G_4 can be express as products of a, b, c and their inverses. Let’s define the following quantities:

$$b_1 = (b^{\frac{p-1}{2}})^{a^{k_1}}, \quad b_3 = b^{a^{k_3}}, \quad b_4 = b^{a^{k_4}} \quad (6)$$

for some integers k_1, k_3, k_4 . Then we can write

$$\begin{aligned} G_1 &= \langle c, b_1 \rangle, & G_2 &= \langle a, b \rangle, & G_3 &= \langle b_1 c^2, b_3 \rangle, \\ G_4 &= \langle b_1 c, b_4 \rangle, & G &= \langle a, b, c \rangle. \end{aligned} \quad (7)$$

As $G_1 \cong D_6$, we should have the relation $c^3 = (cb_1)^2 = 1$. For G_3 and G_4 to be dihedral groups, we need $(b_3 b_1 c^2)^2 = (b_4 b_1 c)^2 = 1$. We may use GAP to determine a concrete group with these relations, but there are still too many parameters to choose and we do not know which ones may yield the correct structure.

Observe in the original violation, the structure $G_4 = G_3^{(3, 4, 5)}$ with generators $(1, 3, 4, 2)^{(3, 4, 5)} = (1, 2, 5, 4)^{-1}$ is not utilized yet. If we let $a = (1, 2, 3, 4, 5)$, $b = (1, 4, 3, 5)$, $c = (3, 4, 5)$, $b_3 = (1, 3, 4, 2)$, $b_4 = (1, 2, 5, 4)$ in the original construction, then the relation above translates to $b_3^c = b_4^{-1}$. We claim this relation for our presentation automatically makes $(b_3 b_1 c^2)^2 = (b_4 b_1 c)^2 = 1$ if and only if $k_3 - k_1 \equiv k_1 - k_4 \pmod{p}$: as $|b_1| = 2$, $c^3 = (cb_1)^2 = 1 \Rightarrow cb_1 = b_1 c^2$, $(b_3 b_1 c^2)^2 = b_3 b_1 c^{-1} b_3 c b_1 = b_3 b_1 b_4^{-1} b_1$ by the new relation. Similarly $(b_4 b_1 c)^2 = b_4 b_1 b_3^{-1} b_1 = ((b_3 b_1 c^2)^{-2})^{b_1}$, so $(b_3 b_1 c^2)^2 = 1 \Leftrightarrow (b_4 b_1 c)^2 = 1$. Plugging in (6) and using $(a^i)^{b^k} = a^{is^k}$ we have

$$(b_3 b_1 c^2)^2 = a^{[(k_3 - k_1) + (k_1 - k_4)s^{(p-1)/2}](s^{-1} - 1)}.$$

Since s is a primitive root modulo p , $|s^{(p-1)/2}| = 2$. As \mathbb{Z}_p^\times is cyclic of an even order $p-1$, it is clear that there is a unique element of order 2. Also $|(p-1)| = 2$ as $(p-1)^2 \equiv 1 \pmod{p}$, so $s^{(p-1)/2} = p-1$, and

$$(b_3 b_1 c^2)^2 = a^{[(k_3 - k_1) - (k_1 - k_4)](s^{-1} - 1)}.$$

Now $p \nmid (s^{-1} - 1)$ as $|s| = p - 1$, then $(b_3 b_1 c^2)^2 = 1$ if and only if $p \mid [(k_3 - k_1) - (k_1 - k_4)]$ if and only if $k_3 - k_1 \equiv k_1 - k_4 \pmod{p}$. This condition gives us a smaller set of parameters as well as a simpler presentation, while maintains all the structures of the subgroups. (Actually once $k_3 - k_1 \equiv k_1 - k_4 \pmod{p}$ is satisfied, it is very easy to use GAP to produce the desired structures, even with arbitrary k_1 and k_3 .)

In sum, our analysis gives us the following presentation:

$$G = \left\langle a, b, c \mid \begin{array}{l} a^p = b^{p-1} = c^3 = 1, a^b = a^s, \\ (cb_1)^2 = b_3 b_4 = 1 \end{array} \right\rangle \quad (8)$$

where p is an odd prime, s is a primitive root modulo p , $k_3 - k_1 \equiv k_1 - k_4 \pmod{p}$. If our extension of the subgroup structures succeeds, then the orders of subgroups and intersections would be: $|G_1| = 6$, $|G_2| = p(p-1)$, $|G_3| = |G_4| = 2(p-1)$, $|G_{12}| = |G_{13}| = |G_{14}| = 2$, $|G_{23}| = |G_{24}| = p-1$, $|G_{34}| = |G_{123}| = |G_{124}| = 1$. *LHS* of (3) = $6p(p-1)$ while *RHS* = $8(p-1)^2$. So for $p \geq 5$, Ingelton should be violated.

V. EXPLICIT VIOLATION CONSTRUCTION WITH $PGL(2, p)$

Plugging the above presentation into GAP with different p 's and other parameters, we get a series of groups. However, when p is large, GAP usually runs out of memory for some (even simple) operations. According to our computation, for $p = 5, 7, \dots, 23$ GAP determined that they are all finite groups and all violate Ingelton. Among these groups GAP determined their isomorphism types up to $p = 19$, most of which are semidirect products $PSL(2, p) \rtimes \mathbb{Z}_2$. As $PGL(2, p)$'s are also semidirect products of $PSL(2, p)$ and \mathbb{Z}_2 , and $PGL(2, 5) \cong S_5$, we guess the isomorphism type for these groups might just be $PGL(2, p)$. This conjecture is verified by GAP up to $p = 11$.

Although $PGL(2, p)$'s are relatively easy groups of matrices, GAP uses isomorphic permutation groups to represent them. This makes it difficult to recognize the corresponding matrices of the output subgroups. However, with presentation (8) we may explicitly identify the generators in $PGL(2, p)$ and check their relations, then use (7) to construct the subgroups.

Let p be an odd prime. For $A \in GL(2, p)$, let \overline{A} denote the left coset of A in $GL(2, p)$ with respect to $V = \{\alpha I : \alpha \in \mathbb{F}_p^\times\}$. Thus $\overline{A} = \overline{B}$ if and only if each entry of A is a nonzero constant multiple of the corresponding entry of B . We denote the elements of \mathbb{F}_p by ordinary integers, but the addition and multiplication, as well as equality, are modulo p . Furthermore, $-k$ and k^{-1} denotes the additive and multiplicative inverses of k in \mathbb{F}_p respectively. This would not cause any confusion as we only use elements from \mathbb{F}_p in the entries of matrices.

Consider the following matrices in $GL(2, p)$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}, \quad C = \begin{bmatrix} 1 & \frac{p-1}{2} \\ 2 & 0 \end{bmatrix} \quad (9)$$

where t is a primitive root modulo p , i.e. a generator of \mathbb{F}_p^\times . Our guess is $\overline{A}, \overline{B}, \overline{C}$ corresponds to the generators a, b, c in

(8) respectively. The powers of these matrices are:

$$A^k = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \quad B^k = \begin{bmatrix} 1 & 0 \\ 0 & t^k \end{bmatrix},$$

$$C^2 = \begin{bmatrix} 0 & \frac{p-1}{2} \\ 2 & -1 \end{bmatrix}, \quad C^3 = \begin{bmatrix} p-1 & 0 \\ 0 & p-1 \end{bmatrix}$$

for any integer k . Thus $\overline{A}^p = \overline{I}$, $\overline{B}^{p-1} = \overline{I}$, $\overline{C}^3 = \overline{I}$ and $|\overline{A}| = p$, $|\overline{B}| = p-1$, $|\overline{C}| = 3$. Also

$$A^B = B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ t^{-1} & 1 \end{bmatrix} = A^s,$$

where $s = t^{-1}$ is also a primitive root modulo p . So $\overline{A}^{\overline{B}} = \overline{A}^s$. Next we let

$$B_1 = (B^{\frac{p-1}{2}})^{A^{k_1}} = A^{-k_1} B^{\frac{p-1}{2}} A^{k_1}$$

$$= \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -2k_1 & -1 \end{bmatrix},$$

where we calculated $t^{\frac{p-1}{2}} = -1$ as it is the unique element of order 2 in \mathbb{F}_p^\times . Now check

$$CB_1 = \begin{bmatrix} 1 & \frac{p-1}{2} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2k_1 & -1 \end{bmatrix} = \begin{bmatrix} 1+k_1 & \frac{p+1}{2} \\ 2 & 0 \end{bmatrix},$$

$$(CB_1)^2 = \begin{bmatrix} 1+k_1 & \frac{p+1}{2} \\ 2 & 0 \end{bmatrix}^2$$

$$= \begin{bmatrix} (1+k_1)^2 + 1 & (1+k_1)\frac{p+1}{2} \\ 2(1+k_1) & 1 \end{bmatrix}.$$

Thus if we want $(\overline{CB_1})^2 = \overline{I}$, k_1 must be -1 . In this case

$$\overline{B_1} = \overline{\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}}, \quad \overline{CB_1} = \overline{\begin{bmatrix} 0 & \frac{p+1}{2} \\ 2 & 0 \end{bmatrix}} = \overline{\begin{bmatrix} 0 & \frac{p-1}{2} \\ -2 & 0 \end{bmatrix}},$$

$$(\overline{CB_1})^2 = \overline{I}.$$

Next we calculate:

$$B^{A^k} = A^{-k} B A^k = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ k(t-1) & t \end{bmatrix}.$$

Let $B_3 = B^{A^{k_3}}$, $B_4 = B^{A^{k_4}}$. As $k_1 = -1$, $k_3 - k_1 = k_1 - k_4$, we have $k_4 = -2 - k_3$.

$$B_3 C B_4 = \begin{bmatrix} 1 & 0 \\ k_3(t-1) & t \end{bmatrix} \begin{bmatrix} 1 & \frac{p-1}{2} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k_4(t-1) & t \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{p-1}{2} \\ k_3(t-1) + 2t & k_3(t-1)\frac{p-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k_4(t-1) & t \end{bmatrix},$$

whose (2, 2)-entry is $k_3(t-1)\frac{p-1}{2}t$. If we want $(\overline{B_3})^{\overline{C}} \cdot \overline{B_4} = \overline{I} \Leftrightarrow \overline{B_3 C B_4} = \overline{I}$, k_3 must be 0 as the (2, 2)-entry of C is 0 and all $t-1$, $\frac{p-1}{2}$, t are nonzero. So $k_4 = -2 - k_3 = -2$,

$$\overline{B_3} = \overline{\begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}} = \overline{B}, \quad \overline{B_4} = \overline{\begin{bmatrix} 1 & 0 \\ 2(1-t) & t \end{bmatrix}},$$

$$\overline{B_3 C B_4} = \overline{\begin{bmatrix} 1 & \frac{p-1}{2} \\ 2t & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2(1-t) & t \end{bmatrix}} = \overline{\begin{bmatrix} t & \frac{p-1}{2}t \\ 2t & 0 \end{bmatrix}} \\ = \overline{C}.$$

So far for $\overline{A}, \overline{B}, \overline{C}$ we have verified all the relations in (8). We can also prove that they are actually a set of generators for $PGL(2, p)$. Observe that each matrix in $GL(2, p)$ can be written as a product of the following elementary matrices:

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & t^k \end{bmatrix}, \begin{bmatrix} t^k & 0 \\ 0 & 1 \end{bmatrix}$$

which are generated by $A, A^T, B, t^{-1}B$. So $PGL(2, p)$ is generated by $\overline{A}, \overline{A^T}, \overline{B}$. Now as $t^l = -2$ for some integer l , $t^{-l} = (-2)^{-1} = \frac{p-1}{2}$. We have

$$B^{-l} A^{-2} C B^l \\ = \begin{bmatrix} 1 & 0 \\ 0 & \frac{p-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{p-1}{2} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 1 & \frac{p-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A^T,$$

Thus $\overline{A}, \overline{B}, \overline{C}$ also generate $PGL(2, p)$. So if we set $s = t^{-1}$, $k_1 = -1$, $k_3 = 0$, $k_4 = -2$, then $\overline{A}, \overline{B}, \overline{C}$ corresponds to the generators in (8).

Remark 5: Note that we have not proved that (8) is a presentation of $PGL(2, p)$. In order to do that, one must show that for any group generated by a, b, c while satisfying the relations in (8), the order must be no more than $|PGL(2, p)| = (p-1)p(p+1)$. This is not proved yet. However, identifying possible corresponding generators still gives us a way to explicitly construct the subgroups to violate Ingleton.

Now we can write out the subgroups in $PGL(2, p)$ corresponding to subgroups in (7).

$G_1 = \langle \overline{C}, \overline{B_1} \rangle$. Note that $|\overline{C}| = 3$, $|\overline{B_1}| = 2$, $(\overline{C B_1})^2 = \overline{I} \Leftrightarrow \overline{C B_1} = \overline{B_1}(\overline{C})^2$, so G_1 has at most 6 elements $\{(\overline{B_1})^i (\overline{C})^j : 0 \leq i < 2, 0 \leq j < 3\}$. Calculating these elements we can see $|G_1| = 6$ exactly:

$$G_1 = \left\{ \overline{I}, \begin{bmatrix} 1 & \frac{p-1}{2} \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{p-1}{2} \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & \frac{p-1}{2} \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{p-1}{2} \\ -2 & 0 \end{bmatrix} \right\}.$$

$G_2 = \langle \overline{A}, \overline{B} \rangle$. We claim that G_2 is just the subgroup of lower triangular matrices in $GL(2, p)$ modulo V , i.e.

$$G_2 = \left\{ \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix} : \alpha \in \mathbb{F}_p, \beta \in \mathbb{F}_p^\times \right\}.$$

As A, B are lower triangular, any element in G_2 is a lower triangular matrix modulo V . On the other hand, $\forall \alpha \in \mathbb{F}_p, \beta \in \mathbb{F}_p^\times, \beta = t^l$ for some integer l . So

$$\begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix} = A^\alpha B^l \Rightarrow \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix} = \overline{A}^\alpha \overline{B}^l \in G_2.$$

Therefore $|G_2| = p(p-1)$.

$G_3 = \langle \overline{B_1}(\overline{C})^2, \overline{B_3} \rangle = \langle \overline{C B_1}, \overline{B_3} \rangle$. Note that $|\overline{C B_1}| = 2$, $|\overline{B_3}| = |\overline{B}| = p-1$, also

$$\overline{B_3} \cdot \overline{C B_1} = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 0 & \frac{p-1}{2} \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{p-1}{2} \\ -2t & 0 \end{bmatrix} \\ = \begin{bmatrix} 0 & \frac{p-1}{2} t^{-1} \\ -2 & 0 \end{bmatrix} = \overline{C B_1}(\overline{B_3})^{-1},$$

so G_3 has at most $2(p-1)$ elements $\{(\overline{C B_1})^i (\overline{B_3})^j : 0 \leq i < 2, 0 \leq j < p-1\}$. Calculating these elements we can see $|G_3| = 2(p-1)$ exactly:

$$G_3 = \left\{ \begin{array}{l} (\overline{B_3})^k = \begin{bmatrix} 1 & 0 \\ 0 & t^k \end{bmatrix}, \\ \overline{C B_1}(\overline{B_3})^k = \begin{bmatrix} 0 & \frac{p-1}{2} t^k \\ -2 & 0 \end{bmatrix} \end{array} : 0 \leq k < p-1 \right\}.$$

$G_4 = \langle \overline{B_1 C}, \overline{B_4} \rangle$. Note that

$$\overline{B_1 C} = \begin{bmatrix} 1 & \frac{p-1}{2} \\ 0 & -1 \end{bmatrix}, \quad (\overline{B_4})^k = \begin{bmatrix} 1 & 0 \\ 2(1-t^k) & t^k \end{bmatrix},$$

so $|\overline{B_1 C}| = 2$, $|\overline{B_4}| = p-1$. Also

$$\overline{B_4} \cdot \overline{B_1 C} = \begin{bmatrix} 1 & \frac{p-1}{2} \\ 2(1-t) & -1 \end{bmatrix} = \begin{bmatrix} t^{-1} & \frac{p-1}{2} t^{-1} \\ 2(t^{-1}-1) & -t^{-1} \end{bmatrix} \\ = \begin{bmatrix} 1 & \frac{p-1}{2} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2(1-t^{-1}) & t^{-1} \end{bmatrix} = \overline{B_1 C}(\overline{B_4})^{-1},$$

so G_4 has at most $2(p-1)$ elements $\{(\overline{B_1 C})^i (\overline{B_4})^j : 0 \leq i < 2, 0 \leq j < p-1\}$. Calculating these elements we can see $|G_4| = 2(p-1)$ exactly:

$$G_4 = \left\{ \begin{array}{l} (\overline{B_4})^k = \begin{bmatrix} 1 & 0 \\ 2(1-t^k) & t^k \end{bmatrix}, \\ \overline{B_1 C}(\overline{B_4})^k = \begin{bmatrix} 1 & \frac{p-1}{2} \\ 2(1-t^{-k}) & -1 \end{bmatrix} \end{array} : 0 \leq k < p-1 \right\}.$$

With all four subgroups explicitly written, we can easily write down the intersections:

$$G_{12} = \left\{ \overline{I}, \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \right\},$$

$$G_{13} = \left\{ \overline{I}, \begin{bmatrix} 0 & \frac{p-1}{2} \\ -2 & 0 \end{bmatrix} \right\},$$

$$G_{14} = \left\{ \overline{I}, \begin{bmatrix} 1 & \frac{p-1}{2} \\ 0 & -1 \end{bmatrix} \right\},$$

$$|G_{12}| = |G_{13}| = |G_{14}| = 2.$$

$$G_{23} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & t^k \end{bmatrix} : 0 \leq k < p-1 \right\},$$

$$G_{24} = \left\{ \begin{bmatrix} 1 & 0 \\ 2(1-t^k) & t^k \end{bmatrix} : 0 \leq k < p-1 \right\},$$

$$|G_{23}| = |G_{24}| = p-1.$$

$$G_{34} = G_{123} = G_{124} = 1.$$

So in (3), indeed $LHS = |G_1||G_2||G_{34}||G_{123}||G_{124}| = 6p(p-1)$, $RHS = |G_{12}||G_{13}||G_{14}||G_{23}||G_{24}| = 8(p-1)^2$, $LHS - RHS = 2(p-1)(4-p)$. Thus Ingleton is violated when $p \geq 5$, and the subgroup structures of S_5 are exactly reproduced.

VI. CONCLUSION

Using a refined search we found the smallest group to violate the Ingleton inequality to be the 120 element group S_5 . Investigating the detailed structure of the subgroups allowed us to determine that this is an instance of the Ingleton-violating family of groups $PGL(2, p)$ for primes $p \geq 5$. We have begun investigating $PGL(2, p^q)$ groups and conjecture that they violate Ingleton for large enough p and q . Computer search verifies that $PGL(2, 2^2)$ does not violate Ingleton, whereas $PGL(2, 2^3)$ and $PGL(2, 3^2)$ do. Finally, investigating the use of these groups to construct network codes more powerful than linear ones may be a fruitful direction for future work.

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